MSW effect in quantum field theory

Christian Y. Cardall*
Department of Physics & Astronomy, State University of New York at Stony Brook, Stony Brook, New York 11794-3800

Daniel J. H. Chung†
Randall Physics Laboratory, University of Michigan, Ann Arbor, Michigan 48109-1120
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We show in detail the general relationship between the Schrödinger equation approach to calculating the MSW effect and the quantum field-theoretical *S*-matrix approach. We show the precise form a generic neutrino propagator must have to allow a physically meaningful ''oscillation probability'' to be decoupled from neutrino production fluxes and detection cross sections, and explicitly list the conditions—not realized in cases of current experimental interest—in which the field theory approach would be useful. [S0556-2821(99)01619-7]

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I. INTRODUCTION

Recent results from the Super-Kamiokande experiment [1] have confirmed, with high statistics, the reality of the solar [2] and atmospheric [3] neutrino anomalies. In the case of the atmospheric neutrino anomaly, the Super-Kamiokande data can be interpreted as providing evidence that the anomaly is due to neutrino flavor mixing [4], and ''long baseline'' accelerator and reactor neutrino experiments may be able to confirm this conclusion [5]. Data from Super-Kamiokande and the Sudbury Neutrino Observatory (SNO) eventually should allow tests to determine if the solar neutrino anomaly is also due to neutrino oscillations [6]. These developments are bringing to a critical test the possibility suggested by earlier observations of solar and atmospheric neutrinos—and by a persistent signal in an accelerator neutrino experiment, the Liquid Scintillation Neutrino Detector (LSND) [7]—that neutrino flavor mixing may provide one of the first experimental windows on physics beyond the standard model.

In a generic neutrino oscillation experiment the event rate for the detection of ν_β from a flux of ν_α from a point source at distance L takes a form such as the following:

$$d\Gamma_{\alpha\beta}=\int dE_{\mathbf{q}}\left(\frac{d\Gamma_{\alpha,\nu_\alpha}}{L^2\,d\Omega_{\mathbf{q}}\,dE_{\mathbf{q}}}\right)(P_{\nu_\alpha\to\nu_\beta})(d\sigma_{\nu_\beta,\beta}), \quad (1)$$

where the direction of the neutrino momentum $\mathbf{q}$ points from the source to the detector. The first factor in the integrand represents the flux of neutrinos of energy $E_{\mathbf{q}}$ from a process involving a charged lepton of flavor α, and the third factor is the cross section for neutrino detection via a process involving a charged lepton of flavor β. These factors are computed by the standard techniques of quantum field theory (QFT), with the approximation of massless neutrinos.

The middle factor—the so called ''oscillation probability''—is typically computed with a quantum mechanical model of the oscillation process. In this model the neutrino state inhabits a Hilbert space whose dimension is equal to the number of neutrino flavors. The Hilbert space is spanned by a mass basis and a flavor basis. These bases are connected by a mixing matrix, taken to be the same as that which connects neutrino ''flavor fields'' to neutrino ''mass eigenstate fields'' in a field theory Lagrangian. The Hamiltonian is simply the particle energy, and the phenomenon of neutrino flavor oscillations arises because the Hamiltonian is not diagonal in the flavor basis. It was pointed out by Wolfenstein [8] that the parameters of flavor oscillations are altered in the presence of matter because of the effective mass induced by neutrino forward scattering off the background. The effective mass, which contributes to the Hamiltonian in this quantum mechanical model, can be computed by employing the famous formula relating the index of refraction to the forward scattering amplitude, where this amplitude is computed from QFT using standard interactions. Since the effective mass due to the background is diagonal in the flavor basis, in the limit of slowly varying background density the total Hamiltonian is diagonal in a new basis, the ''instantaneous mass'' basis. Mikheyev and Smirnov [9] subsequently noted that a level crossing of the ''instantaneous neutrino mass eigenstates in matter'' occurs in a background of monotonically varying density. The resulting Mikheyev-Smirnov-Wolfenstein ''(MSW) effect'' constitutes a new mechanism of flavor transformation that allows a parameter space of solution to, for example, the solar neutrino problem, that is very different from that provided by vacuum neutrino oscillations.

The means just described of computing an experimental event rate for neutrino flavor transformations has the virtue of simplicity. However, being somewhat schizophrenic in its amalgamation of quantum field theoretical and quantum mechanical methods, it is not surprising that studies have appeared in which the neutrino production/oscillation/detection is examined as a single process in the context of QFT, with the neutrinos being virtual particles [10–15]. These studies identify the conditions for which the amplitude for this overall process factorizes. However, by not showing the complete relationship between this amplitude and the neutrino production flux and detection cross section, these studies

*Electronic address: Christian.Cardall@sunysb.edu
†Electronic address: djchung@landau.physics.lsa.umich.edu

lack a firm justification (other than recognition from the usual quantum mechanical picture) for calling a particular factor an ''oscillation amplitude.''[1] In addition, previous works employing this ''scattering approach'' to flavor mixing have only considered vacuum flavor oscillations; it is our purpose here to consider the MSW effect in the context of QFT. Our approach is complementary to the work of Ref. [16] where this is derived from QFT within the context of relativistic Wigner functions.

In this scattering approach to neutrino oscillations, the derivation of the usual Schrödinger equations for the MSW effect in the presence of a spatially varying background is a rather trivial consequence of the virtual neutrinos going on shell, since in that case the multiparticle nature of QFT becomes irrelevant. This property was noted in Ref. [11] in the context of vacuum oscillations. If this property is invalid, then the usual quantum mechanical treatment must be modified and the quantum field theoretical treatment becomes useful. Hence, perhaps the most important point of this paper is that the usual quantum mechanical treatment should be modified suitably if one of the following conditions holds: the on-shell neutrino momentum is nonrelativistic, the neutrino production or detection vertices are non-chiral, the wave packets of the external production/detection particles are sensitive to momentum variations of the order of inverse source-detector distance, or the neutrino effective mass splittings (determined by the effective potential including the background matter contributions) are large compared to the spread in the momenta of the external particles.

We begin in Sec. II with a discussion of general neutrino oscillations. While this ground has been partially explored previously, the discussion will serve to clarify some of the physics of the oscillation process and identify the extent to which the neutrino propagator determines the probability of neutrino oscillations. In particular, we do not assume the vacuum propagator at the outset; we find the precise form a generic propagator must take to allow a physically meaningful ''oscillation probability'' to be decoupled from neutrino production fluxes and detection cross sections, and pinpoint the component that gives rise to the oscillation amplitude. In Sec. III, we discuss the effective Lagrangian approximation assumed in our framework; as specific examples we discuss $e^{\pm}$ and neutrino backgrounds, including an outline of how to obtain a self-consistent neutrino background in, for example, the supernova environment. Having identified the Green's function (or propagator) as that which determines the portion of the neutrino oscillation probability that is independent of the production/detection mechanisms, in Sec. IV we study the Green's function of an effective theory of neutrinos in a static, uniform background, finding a rich pole structure. Great simplification occurs in the relativistic limit, and we recover the same oscillation amplitude obtained with the usual quantum mechanical model. In Sec. V, we study the Green's function in a nonuniform background potential. Under appropriate conditions we recover the usual Schrödinger-type equation for the oscillation amplitude. Section VI contains concluding remarks. An Appendix contains a quick and easy derivation in QFT of fully normalized neutrino oscillation event rates in the form of Eq. (1), which is justified by the more complete treatment in Sec. II.

[1] A partial connection is made in Ref. [11], where it is shown, for example, how the $1/L^2$ flux factor and on shell momentum space neutrino spinors arise from the vacuum propagator. However, the overall event rate they arrive at by ''heuristic consideration,'' to use their words, contains a normalization constant. The relationship of this normalization constant to the coordinate space external particle spinors they employ is left unspecified.

II. GENERAL ν OSCILLATIONS

For the sake of completeness and to establish the setting of our calculation, we give a general overview of the neutrino oscillation calculation in field theory, going beyond previous works [10–15] in a couple of ways. While earlier studies demonstrated the factorization of the amplitude under suitable conditions—enabling identification of the factor called the ''oscillation amplitude'' in the standard picture—they do not go all the way to a fully normalized expression like that of Eq. (1), leaving one without an unambiguous, physically meaningful reason to name this factor an ''oscillation amplitude.'' In addition, previous works exploring the conditions under which the neutrino propagator determines the oscillation probability, independent of the details of neutrino production and detection, have only considered the vacuum propagator. As a prelude to studying the MSW effect, we seek to elucidate the form a generic neutrino propagator must have to make it possible to disentangle the neutrino oscillation probability from the production/detection mechanisms.

In this section we consider generalities without committing to a particular model (Lagrangian), establishing the connection between a physically clear definition of oscillation probability [given by Eq. (1)] and the neutrino propagator. For a simple choice of external particle wave packets, we will give a fairly general expression for the neutrino oscillation event rate including a ''calculated'' normalization. For illustration, in the Appendix we give an explicit calculation leading to a fully normalized event rate like Eq. (1) in the context of a semirealistic model Lagrangian.

In field theory and in physical situations, we distinguish a given flavor of neutrinos by their interactions with charged leptons. If we calculate the probability amplitude for the process involving flavor α at the source of the neutrinos and flavor β at the detector of the neutrinos, we can calculate the probability of neutrino oscillations of $\alpha \rightarrow \beta$. The flavor of each interaction can be distinguished by measurable, on shell, external particles (i.e., the charged leptons). Hence, calculating neutrino oscillations is equivalent to calculating a scattering event where a neutrino propagator connects two flavor distinguishing vertices with external particles coming from them.

In calculating scattering quantities such as cross sections, we usually calculate the plane wave scattering S matrix:

$$S(\{k_i\},\{p_j\}) - 1 \equiv (2\pi)^4 \delta^4\left(\sum_l (-1)^{d_l} p_l + \sum_l (-1)^{s_l} k_l \right) i\mathcal{M}, \tag{2}$$

where $\mathcal{M}$ is the usual invariant amplitude calculated with Feynman diagrams in momentum space and $s_l = d_l = 1$ for incoming particles and 0 for outgoing particles (the grouping of the momenta will be explained shortly). This is considered to be a good approximation in the case of calculating usual collider event rates since there the events of interest occur within a single volume element before the final state particles are detected, and the corrections arising from localization of the interactions usually are not important to the detection rate. However, because neutrino oscillations involve quantum interference effects over macroscopic distances which separate the production point and the detection point of the neutrinos, we must take more care to account for the localization of the interaction points to calculate the leading order observable quantity.

This localization of the vertices necessarily requires that the incident and final states be spatially localized wave packets instead of plane wave states. (As we shall see, analyses which appear to use only plane wave external states while restricting spatial integrations in an *ad hoc* manner [12,15] actually have complicated wave packets buried beneath the surface.) Hence, the probability amplitude[2] is a superposition of Eq. (2),

$$\mathcal{A} = \int \prod_j^{1+F_D} [dp_j]\psi_{Dj}(p_j, \underline{p}_j) \prod_i^{I_S+F_S} [dk_i]\psi_{Si}(k_i, \underline{k}_i) \times [S(\{k_m\},\{p_m\}) - 1], \quad (3)$$

where $[dp_j] = d^3\mathbf{p}_j/[(2\pi)^3\sqrt{2E_{\mathbf{p}_j}}]$, $\{k_m\}$ are the external momenta of the vertex at the production region centered about $\mathbf{x}_S$, and $\{p_m\}$ are the external momenta of the detection region centered about $\mathbf{y}_D$. Here the set of parameters $\underline{p}_m$ and $\underline{k}_m$ characterize the peak of the wave packets' distribution of momenta.[3] We have also fixed the number of external particles to be I_S incoming and F_S outgoing particles at the source vertex, and 1 incoming and F_D outgoing external particles at the detector vertex.

Now, let us see how this is related to the usual quantum mechanical treatment. The procedure for calculating $P_{\nu_\alpha \to \nu_\beta}$ in Eq. (1) in a quantum mechanical model was described in Sec. I. In field theory, $P_{\nu_\alpha \to \nu_\beta}$ defined by Eq. (1) can be calculated by comparing Eq. (1) with the event rate derived from Eq. (3). We associate $\mathcal{A}_S$ with the amplitude for $\{I_S\} \to \{F_S\} + \overset{(-)}{\nu}_\alpha$ at the source and $\mathcal{A}_D$ with the amplitude for $D + \overset{(-)}{\nu}_\beta \to \{F_D\}$ at the detector (we specialize to one detector particle D), and choose plane wave packets for the source's final state (anti)neutrinos and the detector's initial state (anti)neutrinos:

$$\mathcal{A}_S = \frac{1}{\sqrt{2E_{\mathbf{q}}}\sqrt{V}} \int \prod_i^{I_S+F_S} [dk_i]\psi_{Si}(k_i, \underline{k}_i)[S_S(\{k_m\},q) - 1],$$

$$\mathcal{A}_D = \frac{1}{\sqrt{2E_{\mathbf{q}}}\sqrt{V}} \int \prod_j^{1+F_D} [dp_j]\psi_{Dj}(p_j, \underline{p}_j)[S_D(\{p_m\},q) - 1], \quad (4)$$

where $q = (E_{\mathbf{q}}, \hat{\mathbf{L}}E_{\mathbf{q}})$ is the neutrino momentum, and $\hat{\mathbf{L}} = (\mathbf{y}_D - \mathbf{x}_S)/|\mathbf{y}_D - \mathbf{x}_S|$ points from the source to the detector. In $\mathcal{A}_S$ and $\mathcal{A}_D$ we have implicitly assumed that the neutrinos are massless, because massive flavor eigenstates cannot be asymptotic states. Standard kinematics then yields the relationship

$$\frac{|\mathcal{A}|^2}{T} = \int \frac{dE_{\mathbf{q}} E_{\mathbf{q}}^2}{(2\pi)^3 L^2 v_{\nu D}} \frac{|\mathcal{A}_S|^2 V}{T_S} P_{\overset{(-)}{\nu}_\alpha \to \overset{(-)}{\nu}_\beta} \frac{|\mathcal{A}_D|^2 V}{T_D}, \quad (5)$$

where V is the usual total volume factor associated with the phase space and normalization of plane wave packets; T, T_S, and T_D are the usual time factors associated with stationary wave packets; $v_{\nu D}$ is the Møller speed (associated with the flux) between the detector particle and the neutrinos; and $L \equiv |\mathbf{y}_D - \mathbf{x}_S|$.

As just indicated, we make the simplifying assumption of stationary wave packets. The main simplifying utility of this energy conservation approximation is to get rid of the neutrino momentum integral.[4] We encode our assumption of stationarity by defining spatially smeared functions

$$g_S(\mathbf{x},\{\underline{k}_i\},q) \quad e^{i\sum_l (-1)^{s_l}\underline{k}_l \cdot x} = \int \prod_j^{I_S+F_S} [dk_j]\psi_{Sj}(k_j, \underline{k}_j) e^{i\sum_l (-1)^{s_l} k_l \cdot x} i\mathcal{M}_S(\{k_i\},q),$$

$$g_D(\mathbf{y},\{\underline{p}_i\},q) e^{i\sum_l (-1)^{d_l}\underline{p}_l \cdot y} = \int \prod_j^{1+F_D} [dp_j]\psi_{Dj}(p_j, \underline{p}_j) e^{i\sum_l (-1)^{d_l} p_l \cdot y} i\mathcal{M}_D(\{p_i\},q). \quad (6)$$

We note that $\mathcal{M}_S$ and $\mathcal{M}_D$ have the form (assuming $V-A$ lepton currents)

$$\mathcal{M}_S(\{k_i\},q) = \overline{u^-}(q) P_R M_1(\{k_i\}),$$

$$\mathcal{M}_D(\{p_i\},q) = M_2(\{p_i\}) P_L u^-(q) \quad (\nu \text{ osc}),$$

[2]Our conventions for the metric, γ matrices, and normalizations are the same as Ref. [17]. The wave packet normalization is $\int d^3\mathbf{p}(2\pi)^{-3}|\psi(\mathbf{p})|^2 = 1$. A plane wave packet $\psi(\mathbf{p},\mathbf{p}') = [(2\pi)^3/\sqrt{V}]\delta^3(\mathbf{p}-\mathbf{p}')$, where V is a volume factor, follows this normalization convention provided $[\delta^3(\mathbf{p}-\mathbf{p}')]^2$ is interpreted as $[V/(2\pi)^3]\delta^3(\mathbf{p}-\mathbf{p}')$.

[3]The 0th component of these "parameters" is taken to be on mass shell since we will choose the wave packets such that they behave like plane waves for large values of the spatial components of these "parameters."

[4]We refer the reader to Ref. [14] and references therein for related discussions regarding coherence.

$$\mathcal{M}_S(\{k_i\},q)=M_2(\{k_i\})P_L v^+(q),$$
$$\mathcal{M}_D(\{p_i\},q)=\overline{v^+}(q)P_R M_1(\{p_i\}) \quad (\overline{\nu}\ \text{osc}), \tag{7}$$

where M_1 and M_2 are, respectively, column and row vectors in spinor space, and P_L and P_R are the left- and right-handed chiral projection operators. Following the conventions of [17], we represent the spinors u and v as

$$u^s(q)=\begin{pmatrix}\sqrt{q\cdot\sigma}\xi^s\\ \sqrt{q\cdot\overline{\sigma}}\xi^s\end{pmatrix}, \quad v^s(q)=\begin{pmatrix}\sqrt{q\cdot\sigma}\eta^s\\ -\sqrt{q\cdot\overline{\sigma}}\eta^s\end{pmatrix}, \tag{8}$$

where $\sigma^\mu=(1,\boldsymbol{\sigma}),\overline{\sigma}^\mu=(1,-\boldsymbol{\sigma})$, and $\boldsymbol{\sigma}$ is the three-vector of Pauli matrices. Since these are spinors for massless particles, the spin index s is associated with the spin component along the momentum axis; specifically, we have $\xi^-=\eta^+$, $\xi^+=-\eta^-$, with $\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}\xi^\pm(\hat{\mathbf{q}})=\pm\xi^\pm(\hat{\mathbf{q}})$. Setting Eqs. (4), (6) into Eq. (5) yields

$$(2\pi)\delta\left(\sum_l(-1)^{s_l}E_{\underline{\mathbf{k}}_l}+\sum_l(-1)^{d_l}E_{\underline{\mathbf{p}}_l}\right)P_{\overset{(-)}{\nu}_\alpha\to\overset{(-)}{\nu}_\beta}$$
$$=16\pi^2L^2v_{\nu D}\frac{|\mathcal{A}|^2}{T}|\tilde{g}_S(-\mathbf{q},\{\underline{k_i}\})|^{-2}|\tilde{g}_D(\mathbf{q},\{\underline{p_i}\})|^{-2}, \tag{9}$$

where we have defined

$$\tilde{g}_D(\mathbf{q},\{\underline{p_i}\})\equiv\int d^3\mathbf{y}\,g_D(\mathbf{y},\{\underline{p_i}\},q)e^{i(\mathbf{q}-\sum_l(-1)^{d_l}\underline{\mathbf{p}}_l)\cdot\mathbf{y}}, \tag{10}$$

and similarly for $\tilde{g}_S$. Stationarity constrains $|\mathbf{q}|\equiv E_{\mathbf{q}}=-\sum_l(-1)^{s_l}E_{\underline{\mathbf{k}}_l}$.

Let us turn our attention to $\mathcal{A}$. Given that we have a neutrino propagator G in our amplitude, and assuming $V-A$ lepton currents, we can write

$$S(\{k_i\},\{p_j\})-1$$
$$=\int d^4y\,e^{i\sum_l(-1)^{d_l}p_l\cdot y}\int d^4x\,e^{i\sum_l(-1)^{s_l}k_l\cdot x}$$
$$\times i\int\frac{d^4s}{(2\pi)^4}e^{\mp is\cdot(y-x)}M_2\,P_L\,G(s)\,P_R\,M_1, \tag{11}$$

where M_1 and M_2 are the same as in Eqs. (7), and s is the off-shell propagator momentum. The upper (lower) sign of $\mp$ in the exponential is for neutrino (antineutrino) oscillations; this arises from choosing x (y) to always correspond to the source (detector). That is, for neutrino oscillations of flavor α to flavor β, the Green's function is $iG^{\beta\alpha}(y,x)=\langle T\{\nu^\beta(y)\overline{\nu^\alpha}(x)\}\rangle_0=i\int[d^4s/(2\pi)^4]\,e^{-is\cdot(y-x)}G^{\beta\alpha}(s)$ (with $T\,\{\}$ and $\langle\rangle_0$ denoting a time-ordered product and vacuum expectation value, respectively), while for antineutrino oscillations $\alpha\to\beta$, the labeling is $iG^{\alpha\beta}(x,y)$.

Insert the identity

$$\frac{q^\mu Q^\nu\{\gamma_\mu,\gamma_\nu\}}{2q\cdot Q}=1 \tag{12}$$

on both sides of the Green's function in Eq. (11), where as before $q=(E_{\mathbf{q}},\hat{\mathbf{L}}E_{\mathbf{q}})$, and we define $Q=[E_{\mathbf{q}},\hat{\mathbf{L}}\sqrt{(E_{\mathbf{q}})^2-m^2}]$ in which the parameter $m^2<E_{\mathbf{q}}^2$, though its precise value is unimportant in this context. We note that since q is null, $q\cdot\gamma=\sum_s u^s(q)\overline{u^s}(q)$ (or $q\cdot\gamma=\sum_s v^s(q)\overline{v^s}(q)$, if one wishes to consider antineutrino oscillations). From the explicit form of u and v it is easy to see that

$$P_L\,(q\cdot\gamma)(Q\cdot\gamma)=P_L\,u^-(q)\overline{u^-}(q)\,(Q\cdot\gamma),$$
$$P_L\,(Q\cdot\gamma)(q\cdot\gamma)=(Q\cdot\gamma)\,P_R\,u^+(q)\overline{u^+}(q),$$
$$(Q\cdot\gamma)(q\cdot\gamma)\,P_R=(Q\cdot\gamma)\,u^-(q)\overline{u^-}(q)\,P_R,$$
$$(q\cdot\gamma)(Q\cdot\gamma)\,P_R=u^+(q)\overline{u^+}(q)\,P_L\,(Q\cdot\gamma). \tag{13}$$

[The same relations hold for $u^\pm(q)$ replaced by $v^\mp(q)$.] Soon we will show the form that the Green's function must have, after localization by the source and detector, in order that the term with u^- (or v^+) on both sides of G be the only one to contribute. If more than one spin contributes, we will not recover the usual quantum mechanical treatment without spins taken into account. In that case, working with the full scattering picture of Eq. (3) is useful. Keeping only the term with the relevant spinor on either side of G, one can show that Eq. (3) becomes

$$\mathcal{A}=-\int d^4y\,g_D(\mathbf{y},\{\underline{p_i}\},q)\,e^{i\sum_l(-1)^{d_l}\underline{p}_l\cdot y}$$
$$\times\int d^4x\,g_S(\mathbf{x},\{\underline{k_i}\},q)\,e^{i\sum_l(-1)^{s_l}\underline{k}_l\cdot x}$$
$$\times i\int\frac{d^4s}{(2\pi)^4}e^{\mp is\cdot(y-x)}\overline{P}\,G(s)\,P, \tag{14}$$

where $P=\gamma^0u^-(q)/(2E_{\mathbf{q}})=\gamma^0v^+(q)/(2E_{\mathbf{q}})$, $\overline{P}=P^\dagger\gamma^0$, and g_S and g_D are given by Eq. (6).

In passing, we would like to remark that in Eq. (14), we can always write (for neutrino oscillations, for example)

$$g_S(\mathbf{x},\{\underline{k_i}\},q)\,e^{i\sum_l(-1)^{s_l}\underline{k}_l\cdot x}=i\overline{u^-}(q)\,M_1(\{\underline{k_i}\})\,f_S(x,\{\underline{k_i}\}) \tag{15}$$

(and similarly for g_D) where f_S is a scalar function of $\mathbf{x}$. Then we obtain the form (again assuming the single-spin contribution is justified after spatial integration)

$$\mathcal{A}=\int d^4x\,d^4y\,f_S(x,\{\underline{k_i}\})f_D(y,\{\underline{p_i}\})$$
$$\times\int\frac{d^4s}{(2\pi)^4}e^{-is\cdot(y-x)}i\mathcal{M}(\{\underline{k_i}\},s,\{\underline{p_i}\}), \tag{16}$$

which is a common starting point of analysis in the literature as in Refs. [12,15]. However, with arbitrarily chosen smearing functions *f*, it is difficult to assess what actual scattering question the amplitude is an answer to, because the smearing functions are not the wave functions of the in-out particles, but are the wave functions smeared over the matrix elements. As a consequence, the normalization is usually ignored in this approach. We will return to the normalization later in this section.

Returning to Eq. (14), integration over x^0, y^0, and s^0 gives an overall energy-conserving δ function and sets $s^0 = E_{\mathbf{q}}$ (for antineutrino oscillations, $s^0 = -E_{\mathbf{q}}$). We also note that the chiral structure of $\overline{P}GP$ (as well as the original matrix element) picks out only the G_{LR} block of the neutrino propagator, where G_{LR} is the nonzero 2×2 submatrix left by $P_L G P_R$. In addition, the localization of g_S and g_D around $\mathbf{x}_S$ and $\mathbf{y}_D$, respectively, ''clamps down'' on the coordinate space Green's function. In particular, if the characteristic widths L_S and L_D of g_S and g_D are much smaller than the source-detector distance $L = |\mathbf{y}_D - \mathbf{x}_S|$, we note that if the ''oscillation probability'' is to be disentangled from the details of neutrino production and detection, the relevant portion of the Green's function for oscillations $\alpha \to \beta$ must take the form

$$
\begin{aligned}
G_{LR}^{\beta\alpha}(s^0 = E_{\mathbf{q}}, \mathbf{y}, \mathbf{x}) &= \int \frac{d^3\mathbf{s}}{(2\pi)^3} e^{i\mathbf{s}\cdot(\mathbf{y}-\mathbf{x})} G_{LR}^{\beta\alpha}(s^0 = E_{\mathbf{q}}, \mathbf{s}) \\
&\simeq -E_{\mathbf{q}}(1 - \boldsymbol{\sigma}\cdot\hat{\mathbf{L}}) \\
&\quad \times \frac{e^{iE_{\mathbf{q}}\hat{\mathbf{L}}\cdot(\mathbf{y}-\mathbf{x})}}{4\pi|\mathbf{y}_D - \mathbf{x}_S|} H^{\beta\alpha}(E_{\mathbf{q}}, \mathbf{y}_D, \mathbf{x}_S) \quad (\nu \text{ osc}), \\
G_{LR}^{\alpha\beta}(s^0 = -E_{\mathbf{q}}, \mathbf{x}, \mathbf{y}) &\simeq +E_{\mathbf{q}}(1 - \boldsymbol{\sigma}\cdot\hat{\mathbf{L}}) \\
&\quad \times \frac{e^{iE_{\mathbf{q}}\hat{\mathbf{L}}\cdot(\mathbf{y}-\mathbf{x})}}{4\pi|\mathbf{y}_D - \mathbf{x}_S|} \overline{H}^{\alpha\beta}(E_{\mathbf{q}}, \mathbf{y}_D, \mathbf{x}_S) \quad (\overline{\nu} \text{ osc}),
\end{aligned} \tag{17}
$$

where $\hat{\mathbf{L}} = (\mathbf{y}_D - \mathbf{x}_S)/|\mathbf{y}_D - \mathbf{x}_S|$ points from the source towards the detector, and the quantities H and $\overline{H}$ have only flavor indices. The factor $E_{\mathbf{q}}(1 - \boldsymbol{\sigma}\cdot\hat{\mathbf{L}})$ arises from the kinetic term in the Lagrangian, and takes this form due to the relativistic limit. Another key ingredient is the factor $e^{iE_{\mathbf{q}}|\mathbf{y}-\mathbf{x}|}$, which is the leading phase factor in the relativistic limit coming from $e^{i\mathbf{s}\cdot(\mathbf{y}-\mathbf{x})}$ evaluated at the poles of G_{LR}. In addition, $1/|\mathbf{y}-\mathbf{x}|$ comes from the asymptotic expansion of the left-hand side of Eq. (17) in the limit that $|\mathbf{y}-\mathbf{x}| \to \infty$, and it can be considered to be the monopole term in a multipole expansion. We will discuss the validity of the factorization and the asymptotic expansion further below in momentum space.

Before we talk about momentum space, let us give an example of Eq. (17) by considering the vacuum propagator. In that case, it is straightforward to show that (anticipating the relativistic limit)

$$
G_{LR}(s^0, \mathbf{x}, \mathbf{y}) = (s^0 + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})[M^{-1}G_{RR}(s^0, \mathbf{x}, \mathbf{y})], \tag{18}
$$

$$
[M^{-1}G_{RR}(s^0, \mathbf{x}, \mathbf{y})]^{\alpha\beta} = -\frac{e^{i|s^0||\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \sum_j U_{\alpha j} U_{\beta j}^* \times \exp\left(-i\frac{m_j^2|\mathbf{x}-\mathbf{y}|}{2|s^0|}\right), \tag{19}
$$

where G_{RR} is the nonzero 2×2 submatrix left by $P_R G P_R$, M is the mass matrix appearing in the Lagrangian, and the m_j are the mass eigenvalues. In Eq. (19) we have made the flavor indices explicit; the relationship between the flavor fields and mass eigenstate fields is $\nu_\alpha = \Sigma_i U_{\alpha i}\psi_i$, where the $U_{\alpha i}$ are elements of a unitary matrix. For $|s^0|L \gg 1$,

$$
G_{LR}^{\alpha\beta}(s^0, \mathbf{x}, \mathbf{y}) \simeq -\frac{e^{i|s^0||\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}[s^0 - |s^0|\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}(\mathbf{x},\mathbf{y})] \sum_j U_{\alpha j} U_{\beta j}^* \times \exp\left(-i\frac{m_j^2|\mathbf{x}-\mathbf{y}|}{2|s^0|}\right), \tag{20}
$$

where $\hat{\mathbf{r}}(\mathbf{x},\mathbf{y}) = (\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|$. To apply Eq. (20) to neutrino oscillations one takes $s^0 \to E_{\mathbf{q}}$ and $\mathbf{x},\mathbf{y} \to \mathbf{y},\mathbf{x}$. For antineutrino oscillations, $s^0 \to -E_{\mathbf{q}}$ and $\mathbf{x},\mathbf{y} \to \mathbf{x},\mathbf{y}$. After making these substitutions we will be integrating Eq. (20) over localization functions of characteristic widths L_S and L_D centered on $\mathbf{x} = \mathbf{x}_S$ and $\mathbf{y} = \mathbf{y}_D$. This means that for $L_S, L_D \ll L$, we may replace $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{x}_S$ and $\mathbf{y}_D$ everywhere except the phase factors, in which we consider the first order variation

$$
\begin{aligned}
|\mathbf{x}-\mathbf{y}| &\simeq |\mathbf{x}_S - \mathbf{y}_D| + \hat{\mathbf{L}}\cdot[(\mathbf{y}-\mathbf{y}_D) - (\mathbf{x}-\mathbf{x}_S)] \\
&= \hat{\mathbf{L}}\cdot(\mathbf{y}-\mathbf{x}).
\end{aligned} \tag{21}
$$

We see that a necessary mathematical condition (in addition to the relativistic assumption and $E_{\mathbf{q}}L \gg 1$) for Eqs. (18)–(20) to reduce to the form of Eqs. (17) is $m_j^2 L_{S,D}/(2E_{\mathbf{q}}) \ll 1$ [which also implies the more familiar $(m_j^2 - m_i^2) L_{S,D}/(2E_{\mathbf{q}}) \ll 1$]. The physical basis of these conditions can also be inferred from Eqs. (18)–(20). $E_{\mathbf{q}}L \gg 1$ allows the propagating neutrino to become an on-shell relativistic particle, and also allows appreciable oscillation phase to build up over the source-detector distance. $m_j^2 L_{S,D}/(2E_{\mathbf{q}}) \ll 1$ requires that no appreciable oscillation phase build up on length scales comparable to the width of the external particle wave packets. The necessity of these conditions for disentanglement of the flavor oscillations from the details of neutrino production and detection is evident.

The origin of these conditions can also be understood in momentum space. First, rewrite Eq. (14) as

$$
\begin{aligned}
\mathcal{A} = &-(2\pi)\delta\left(\sum_l (-1)^{s_l} E_{\underline{\mathbf{k}}_l} + \sum_l (-1)^{d_l} E_{\underline{\mathbf{p}}_l}\right) \\
&\times i\int \frac{d^3 s}{(2\pi)^3} e^{\pm i\mathbf{s}\cdot(\mathbf{y}_D - \mathbf{x}_S)} \\
&\times e^{-i[\sum_l (-1)^{s_l}\underline{\mathbf{k}}_l\cdot\mathbf{x}_S + \sum_l (-1)^{d_l}\underline{\mathbf{p}}_l\cdot\mathbf{y}_D]} \tilde{h}_D(\mathbf{s}, \{\underline{p}_i\}, q)
\end{aligned}
$$

$$\times \tilde{h}_S(-\mathbf{s},\{\underline{k_i}\},q)\overline{P}\,G\left(s^0=\mp\sum_l (-1)^{s_l}E_{\underline{\mathbf{k}}_l},\mathbf{s}\right)P, \tag{22}$$

where the functions $\tilde{h}_S$ and $\tilde{h}_D$ can be approximated to have no $\mathbf{x}_S$ or $\mathbf{y}_D$ dependence. This can easily be seen to be exactly true for the ideal case of isotropic smearing functions, e.g., $g_D(\mathbf{y},\{\underline{p_i}\},q)=h_D(|\mathbf{y}-\mathbf{y}_D|)$; Eq. (10) then yields $\tilde{g}_D=e^{i\mathbf{u}\cdot\mathbf{y}_D}\tilde{h}_D(|\mathbf{u}|)$, where $\mathbf{u}\equiv\mathbf{s}-\Sigma_l(-1)^{d_l}\underline{\mathbf{p}}_l$. Because the propagator will in general have poles corresponding to the mass of the physical neutrino states, the dominant contribution to the integral in the asymptotic limit $LE_{\mathbf{q}}\to\infty$ will be from a term that contains the integrand of Eq. (22) as a factor evaluated at the poles and stationary phase points (critical points). For the vacuum, the constant potential, and the adiabatically spatially varying background potential cases, one can asymptotically expand the integral Eq. (22) in the limit $LE_{\mathbf{q}}\to\infty$ (similarly as in Ref. [11]) to find that to leading approximation, the term $\tilde{h}_D(\mathbf{s},\{\underline{p_i}\},q)\tilde{h}_S(-\mathbf{s},\{\underline{k_i}\},q)$ can be moved outside of the integral with the replacement $\mathbf{s}\to\mathbf{s}_*$ where $\mathbf{s}_*$ corresponds to one of the critical points. By factoring out $\tilde{h}$, we have implicitly assumed that $\tilde{h}_D(\mathbf{s},\{\underline{p_i}\},q)\tilde{h}_S(-\mathbf{s},\{\underline{k_i}\},q)$ is not sensitive to the splittings in the critical points (otherwise different pole momenta $\mathbf{s}$ of G_{LR} will cause $\tilde{h}_D(\mathbf{s},\{\underline{p_i}\},q)\tilde{h}_S(-\mathbf{s},\{\underline{k_i}\},q)$ to have different values, preventing factorization). This means that the wave packets must be flat in momentum space at least within the range of pole momentum splitting. Also, if this is not the case, one of the poles will not contribute (because the amplitude of the wave packet has fallen off with respect to the amplitude at the other pole), and no neutrino oscillations will occur (or more accurately, the neutrino oscillations will be greatly suppressed relative to the background). We will refer to this flatness of the wave packet as insensitivity to $\mathbf{s}_*$ splitting.

Also note that because of the presence of the exponential in Eq. (22), this leading term in the asymptotic expansion will not be a good approximation unless the inverse "momentum scale height" (i.e., logarithmic derivative) of $\tilde{h}_D\tilde{h}_S$ near the poles is much less than L. Hence, factoring out the wave packet dependence which is crucial for the validity of the usual quantum mechanical treatment requires the wave packet factor $\tilde{h}_D\tilde{h}_S$ to be insensitive under $1/L$ momenta variations as well as the $\mathbf{s}_*$ splitting variations.

While localization is clearly necessary for the observation of oscillations, the source and detector localization scales L_S, L_D implied by Eq. (6) cannot be smaller than the Compton wavelength of the lightest external particles. In the case that all the external particles connected to a given vertex are nonrelativistic, this gives rise to a constraint on the masses of these external particles. To see this, consider the ideal case mentioned above in which $g_D(\mathbf{y},\{\underline{p_i}\},q)\approx h_D(|\mathbf{y}-\mathbf{y}_D|)$. Then $\tilde{g}_D=e^{i\mathbf{u}\cdot\mathbf{y}_D}\tilde{h}_D(|\mathbf{u}|)$, where $\tilde{h}_D(|\mathbf{u}|)$ is damped for $|\mathbf{u}|\equiv|\mathbf{s}_*-\Sigma_l(-1)^{d_l}\underline{\mathbf{p}}_l|$ larger than $1/L_D$. Hence, the $\mathbf{s}_*$ splitting insensitivity condition can be written as

$$\left|\sum_l (-1)^{s_l}\underline{\mathbf{k}}_l+\mathbf{s}_*\right|\ll\frac{1}{L_S}<M_{LS},$$

$$\left|\sum_l (-1)^{d_l}\underline{\mathbf{p}}_l-\mathbf{s}_*\right|\ll\frac{1}{L_D}<M_{LD}, \tag{23}$$

where we have denoted the lightest external particle masses to be M_{LS} and M_{LD} for source and detector, respectively. The critical momentum will be $\mathbf{s}_*\approx\hat{\mathbf{L}}\sqrt{[\Sigma_l(-1)^{s_l}E_{\underline{\mathbf{k}}_l}]^2-\tilde{m}_j^2}$ where $\tilde{m}_j$ is the effective pole mass of the particle. Hence, if $\tilde{m}_j\ll|\Sigma_l(-1)^{s_l}E_{\underline{\mathbf{k}}_l}|$ and the external particles are nonrelativistic, then Eq. (23) can be satisfied only if about equal mass of external particles enter and leave the source/detector vertices. If any of the the external particles connected to a given vertex are sufficiently relativistic, this severe constraint does not arise.

We now show that Green's function must take the form found in Eqs. (17) after being spatially "clamped" by the source and detector if the terms projected by the spinors u^- (or v^+) on both sides of G are to be the only contributions to the amplitude. In Sec. IV, where we study the neutrino propagator in a uniform, static medium, we will find it convenient to identify $\hat{\mathbf{L}}$ of Eq. (17) with the positive third spatial direction. In that case it is straightforward to show, using Eq. (13), that the terms with spinors of different spins on either side of G pick out the off-diagonal spinor space elements of G_{LR}, while the terms with the same spins on both sides pick out the diagonal spinor space elements. The matrix $(1-\sigma^3)$ from Eqs. (17) confirms that only G_{LR}^{22} is nonzero, and therefore only the term with $u^-(q)$ $[v^+(q)]$ on both sides of G survives for the neutrino (antineutrino) oscillations.[5]

Recalling that $\mathbf{q}=\hat{\mathbf{L}}\,E_{\mathbf{q}}$, and upon inserting Eqs. (17) into Eq. (14) and Eq. (14) into Eq. (9), we finally arrive at the neutrino oscillation probability

$$P_{\nu_\alpha\to\nu_\beta}=|H^{\beta\alpha}(E_{\mathbf{q}},\mathbf{y}_D,\mathbf{x}_S)|^2,\quad(\nu\ \text{osc}),$$

$$P_{\bar{\nu}_\alpha\to\bar{\nu}_\beta}=|\overline{H}^{\alpha\beta}(E_{\mathbf{q}},\mathbf{y}_D,\mathbf{x}_S)|^2,\quad(\bar{\nu}\ \text{osc}), \tag{24}$$

where H and $\overline{H}$ are defined by Eqs. (17), and we have assumed that the detector particle D is nonrelativistic such that $v_{\nu D}=1$. With the cancellation of the source and detector wave packets, one can see why employing a separate quantum mechanical model to compute the oscillation probability is possible. [Note that the standard vacuum oscillation probability is recovered here, as is clear from Eqs. (17)–(20)].

We emphasize that the Green's function here is the full propagator in any given theory and we have made no severe assumptions about the nature of the production and detection

---

[5]Note that with $\hat{\mathbf{L}}$ set to the third spatial direction, both $u^-(q)$ and $v^+(q)$ have 4-spinor components $(0,\sqrt{2E_{\mathbf{q}}},0,0)$.

effective vertex.[6] Hence, as expected, the production/detection independent field theoretical effects on neutrino oscillations come from the coordinate space Green's function. What perhaps is less expected is the fact that unless the wave packets of the external particles satisfy specific properties, the transition probability will not just depend on the propagator, but the entire coherent scattering process which neutrino oscillation really is. Such tangled wave packet dependence is discussed for example in Ref. [15].

Before concluding this section, we would like to note that we can easily work out the neutrino oscillation detection rate including the normalization if we assume a particular class of wave packets. Let us define a box wave packet as a configuration such that the superposition integral gives, for each outgoing particle, for example,[7]

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}\psi_S(k,\underline{k})e^{ik\cdot x} \approx e^{iE_{\underline{\mathbf{k}}}x^0}\int \frac{d^3\mathbf{k}}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}}\psi_S(k,\underline{k})e^{-i\mathbf{k}\cdot\mathbf{x}} = Ne^{i\underline{k}\cdot x}B(\mathbf{x}-\mathbf{x}_S), \tag{25}$$

where N is a constant independent of x and $B(\mathbf{z})$ is a function which vanishes if $\mathbf{z}$ is outside of a box centered about the origin with each dimension of length L_S and is 1 everywhere else. The approximation in Eq. (25) is valid for $x^0 \ll (E_{\mathbf{k}}L_S)/(2\pi|\mathbf{k}|)$. The normalization condition $\int[d^3\bar{\mathbf{k}}/(2\pi)^3]|\bar{\psi}(k)|^2=1$ fixes N and implies

$$\psi_S(k)=\frac{1}{\sqrt{V}}\left[1-\mathcal{O}\left(\frac{1}{L_S^2E_{\underline{\mathbf{k}}}^2}\right)\right]e^{i(\mathbf{k}-\underline{\mathbf{k}})\cdot\mathbf{x}_S}D_S(\mathbf{k}-\underline{\mathbf{k}}), \tag{26}$$

where

$$D_S(\mathbf{v})=8\frac{\sin(v_xL_S/2)\sin(v_yL_S/2)\sin(v_zL_S/2)}{v_xv_yv_z}. \tag{27}$$

Hence, since the box scale must be larger than the Compton wavelength scale, these external particles will generally have "plane wave in a box" type of normalization [up to a $(2\pi)^3\delta^3(\mathbf{k}-\underline{\mathbf{k}})$ type of localization factor D_S]. This wave packet can be used to calculate the event rate using Eq. (3) in a standard way. Since ψ_S will have a width $2\pi/L_S$, smearing of any function that is proportional to momenta whose magnitude at the peak of the distribution is of the order $2\pi/L_S$, (or less) will deviate significantly from the Dirac δ smearing of that function. Fortunately, because M_1 and M_2 do not depend on such small momenta, we can write the amplitude in the form of Eq. (16) with

$$f_S(x,\{\underline{k}_i\})=\prod_j^{I_S+F_S}\left[\frac{e^{-i(-1)^{s_j}\underline{k}_j\cdot x}}{\sqrt{2E_{\underline{\mathbf{k}}_j}V_S}}B(\mathbf{x}-\mathbf{x}_S)\right], \tag{28}$$

(and similarly for f_D) which is what one would use to calculate scattering of particles confined to a box interacting with particles that can propagate outside of the box. Explicitly, the transition rate per source and detector particle is given by

$$d\Gamma=(2\pi)\,\delta\left(\sum_l(-1)^{s_l}E_{\underline{\mathbf{k}}_l}+\sum_l(-1)^{d_l}E_{\underline{\mathbf{p}}_l}\right)\prod_j^{I_S+F_S}\frac{1}{2E_{\underline{\mathbf{k}}_j}V_S}\times\prod_i^{F_D+1}\frac{1}{2E_{\underline{\mathbf{p}}_i}V_D}\prod_b^{F_S}\frac{d^3\underline{\mathbf{k}}_bV_S}{(2\pi)^3}\prod_a^{F_D}\frac{d^3\underline{\mathbf{p}}_aV_D}{(2\pi)^3}\times\left|\int\frac{d^3\mathbf{s}}{(2\pi)^3}e^{\pm i\mathbf{s}\cdot(\mathbf{y}_D-\mathbf{x}_S)}D_S\left(\sum_l(-1)^{s_l}\underline{\mathbf{k}}_l\pm\mathbf{s}\right)\times D_D\left(\sum_l(-1)^{d_l}\underline{\mathbf{p}}_l\mp\mathbf{s}\right)i\mathcal{M}\right|^2, \tag{29}$$

where D_S is defined by Eq. (27), and D_D is similarly defined. In this case, one can also use the usual heuristic box quantization formalism to calculate the event rates including the normalization. For pedagogical purposes we carry out this simple exercise explicitly using a fermion field toy model in the Appendix.

To summarize this section, we have shown to what extent the neutrino "oscillation probability" is determined by the production/detection wave packet-independent propagator of the field theory. If wave packets for the production and detection events described in Eq. (1) satisfy suitable localization properties and the effective mass splitting of the neutrinos is not large compared to the momentum width of the wave packets, the neutrino propagator determines the probability of transition as defined by Eq. (1). This factoring of the wave packets out of the transition amplitude is crucial to recover the usual quantum mechanical picture of neutrino oscillations. Furthermore, we see how the multiparticle nature of the field theory becomes irrelevant as the poles of the propagator are the only states to contribute in this limit. For this factorization to be possible, the source-detector separation L must be large enough such that the wave packets do not vary over $1/L$ momentum perturbations about the pole momentum, and the pole momenta splitting must be small enough such that the wave packet amplitudes take on approximately the same value for the various pole momenta [as discussed between Eqs. (22) and (23)]. Furthermore, since the usual quantum mechanical treatment neglects the spin of the neutrinos, only one spin projection of the Green function must contribute to the amplitude to recover the usual treatment. We have seen in this section that the relativistic limit

[6]The most significant assumptions leading to our final result were the $V-A$ type lepton currents; the stationary approximation in Eq. (6); relativistic neutrinos, i.e. $(E_{\mathbf{q}}-|\mathbf{s}_*|)/E_{\mathbf{q}}\ll 1$, where $|\mathbf{s}_*|$ is the magnitude of a pole in the momentum space propagator; sufficiently localized and separated source and detector, i.e. $(E_{\mathbf{q}}-|\mathbf{s}_*|)L_{S,D}\ll 1$; and $E_{\mathbf{q}}L\gg 1$.

[7]We will write the wave packet centered about $\mathbf{x}_S$, since the one centered about $\mathbf{y}_D$ is analogous.

of the on shell neutrinos and the chiral nature of the interactions ensure this. Now that we see that wave packet dependence can be factored out (as is implicit in the usual simple quantum mechanical treatment), we shall concentrate on the wave packet-independent field theoretic calculation of the MSW effect, which is encoded in the propagator within a background medium.

III. EFFECTIVE LAGRANGIAN

In this section, we briefly explain the effective potential employed in our calculation of the MSW effect. Focusing on the physics well below the electroweak scale, we write the usual electroweak effective Hamiltonian density as (see, for example, [18])

$$\mathcal{H}_I=\frac{G_F}{\sqrt{2}}(J_c^{\mu}J_{c\mu}^{\dagger}+J_N^{\mu}J_{N\mu}), \tag{30}$$

where J_c^{μ} is the charged current and J_N^{μ} is the neutral current. Take for example the contribution to the neutrino-electron interaction of the form

$$\mathcal{H}_I=\frac{4G_F}{\sqrt{2}}\overline{\nu}_e\gamma^{\mu}P_L\nu_e\overline{e}\gamma_{\mu}P_Le, \tag{31}$$

which will be dominant for the MSW effect. We can distinguish two different types of scattering: forward scattering, for which the background particles do not change their momenta, and non-forward scattering. In calculating our transition rate, we will not account for non-forward scattering contributions because these can be considered to be separate production events. With this restriction, in expanding the S-matrix perturbatively, the main background contribution will come from the expectation values of Eq. (31) taken with respect to the electron background states. The scattering amplitude will then receive contributions proportional to powers of

$$\langle V_{\mu}^{e}\rangle\equiv 2\sqrt{2}G_F\langle n|\overline{e}\gamma_{\mu}P_Le|n\rangle \tag{32}$$

where n labels a many-body background electron state (not necessarily translationally invariant). Note that the right-hand side is proportional to the left-handed electron current of state $|n\rangle$. In an experimental setting, we are really interested in ensemble averages of the probabilities (not the averages of the S matrix). However, for macroscopic numbers of electrons, we expect the main contribution to come from a set of degenerate states having the same spatial localization as the macroscopic distribution function. This approximation will break down if the density matrix is not sharply peaked about one set of states giving degenerate contributions to the scattering amplitude. We will assume that such a peaked distribution exists, and we will merely assign macroscopic currents to the expectation value of currents that will arise in the scattering amplitude calculation. This means that we will replace the interaction Hamiltonian density of Eq. (31) with the effective density

$$\mathcal{H}_I^{\mathrm{eff}}=\frac{4G_F}{\sqrt{2}}\overline{\nu}_e\gamma^{\mu}P_L\nu_eJ_{\mu}^{e}, \tag{33}$$

where J_{μ}^{e} is the macroscopic left-handed electron current. For example, for an unpolarized $e^{\pm}$ background one would employ—based on consideration of the sum over spin states of single particle expectation values of $\overline{e}\gamma^{\mu}P_Le$, for example—the following expression:

$$J_{\mu}^{e}=\frac{1}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}[f_{e^-}(\mathbf{p})-f_{e^+}(\mathbf{p})]\frac{p_{\mu}}{E_{\mathbf{p}}}, \tag{34}$$

where the $f_{e^{\pm}}(\mathbf{p})$ are the usual distribution functions, including a factor of 2 for spin degeneracy. As usual, this procedure neglects higher order correlations. Note that for electrons in thermal equilibrium, our prescription, e.g.,

$$\langle V_{\mu}^{e}\rangle=\sqrt{2}G_F(n_{e^-}-n_{e^+})\delta_{\mu 0} \tag{35}$$

gives the same mass shift as the real time thermal field formalism employed by [19].

As another example, the effective potential due to background neutrinos is of interest in the envelope of a supernova/nascent neutron star, where the neutrino flavor composition can affect, for example, the explosion mechanism [20] or the outcome of possible heavy element nucleosynthesis [21,22]. In addition to the $e^{\pm}$ background, we must consider neutrino-neutrino forward scattering arising from another term in Eq. (30),

$$\mathcal{H}_I=\frac{G_F}{\sqrt{2}}\sum_{i,j}\overline{\nu}_i\gamma^{\mu}P_L\nu_i\overline{\nu}_j\gamma_{\mu}P_L\nu_j, \tag{36}$$

where the indices i,j label the mass eigenstate fields. We work in the mass basis because the external neutrino background consists of on shell states, a point whose consequences were emphasized in Ref. [23] (see also Ref. [22], and references in these). As seen previously, in the perturbative expansion of the S matrix we will have occasion to take a background expectation value of this interaction (this time with respect to a many-body background neutrino state). Two of the neutrino fields will be paired with fields in the "production" and "detection" interactions, leaving two other fields whose background expectation value is taken:

$$\langle\mathcal{H}_I\rangle=\frac{G_F}{\sqrt{2}}\sum_{i,j}2\overline{\nu}_i\gamma^{\mu}P_L\nu_i\langle\overline{\nu}_j\gamma_{\mu}P_L\nu_j\rangle+\frac{G_F}{\sqrt{2}}\sum_{i,j}2\overline{\nu}_i\gamma^{\mu}P_L\langle\nu_i\overline{\nu}_j\rangle\gamma_{\mu}P_L\nu_j. \tag{37}$$

While the correspondence of the expectation value in the first term of Eq. (37) with a macroscopic current is apparent, the meaning of the second term is less clear.

We seek guidance by considering the expectation values with respect to the single particle neutrino states $|\mathbf{q}\, s\, \nu_k\rangle$ of momentum $\mathbf{q}$, spin s, and mass m_k. The expectation value in the first term of Eq. (37) is

$$\langle \mathbf{q}\, s\, \nu_k | \overline{\nu_j} \gamma^\mu P_L \nu_j | \mathbf{q}\, s\, \nu_k \rangle = \frac{\delta_{jk}}{(2\pi)^3 (2E_\mathbf{q})} \overline{u}(\mathbf{q}\, s\, \nu_k) \gamma^\mu P_L u(\mathbf{q}\, s\, \nu_k). \tag{38}$$

We consider a relativistic neutrino background, so that to leading order there are only negative helicity states; then the momentum space spinors in Eq. (38) are approximately

$$u(\mathbf{q}\, s\, \nu_k) \simeq \begin{pmatrix} \sqrt{2E_\mathbf{q}} \xi^-(\hat{\mathbf{q}}) \\ 0 \end{pmatrix}, \quad \xi^-(\hat{\mathbf{q}}) = \begin{pmatrix} -\sin(\theta/2) e^{-i\phi} \\ \cos(\theta/2) \end{pmatrix}, \tag{39}$$

where (θ, ϕ) denote the polar and azimuthal angles that define $\hat{\mathbf{q}}$. The first term in Eq. (37) becomes

$$\frac{G_F}{\sqrt{2}} \sum_{i,j} 2 \overline{\nu_i} \gamma^\mu P_L \nu_i \langle \overline{\nu_j} \gamma_\mu P_L \nu_j \rangle = \sqrt{2} G_F \sum_{i,j} \chi_i^\dagger \frac{\delta_{jk}}{(2\pi)^3} \frac{q_\mu \overline{\sigma}^\mu}{E_\mathbf{q}} \chi_i, \tag{40}$$

where χ_i denotes the upper two components of $P_L \nu_i$ and $\overline{\sigma}^\mu = (1, -\boldsymbol{\sigma})$. Turning to the second term in Eq. (37), one finds

$$\langle \mathbf{q}\, s\, \nu_k | \nu_i \overline{\nu_j} | \mathbf{q}\, s\, \nu_k \rangle = -\frac{\delta_{ik}\delta_{jk}}{(2\pi)^3 (2E_\mathbf{q})} u(\mathbf{q}\, s\, \nu_k) \overline{u}(\mathbf{q}\, s\, \nu_k). \tag{41}$$

Employing Eq. (39), the second term in Eq. (37) becomes

$$\frac{G_F}{\sqrt{2}} \sum_{i,j} 2 \overline{\nu_i} \gamma^\mu P_L \langle \nu_i \overline{\nu_j} \rangle \gamma_\mu P_L \nu_j = \sqrt{2} G_F \chi_k^\dagger \frac{\delta_{ik}\delta_{jk}}{(2\pi)^3} \frac{q_\mu \overline{\sigma}^\mu}{E_\mathbf{q}} \chi_k. \tag{42}$$

Noting the similarity between Eqs. (40) and (42), in the relativistic limit we replace the Hamiltonian density of Eq. (36) with the effective density

$$\mathcal{H}_I^{\rm eff} = \sqrt{2} G_F \sum_{i,j} \overline{\nu_i} J^\mu_{\nu_j} \gamma_\mu P_L \nu_i + \sqrt{2} G_F \sum_i \overline{\nu_i} J^\mu_{\nu_i} \gamma_\mu P_L \nu_i, \tag{43}$$

where

$$J^\mu_{\nu_i} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} [f_{\nu_i}(\mathbf{p}) - f_{\overline{\nu}_i}(\mathbf{p})] \frac{p^\mu}{E_\mathbf{p}}. \tag{44}$$

The flavor fields ν_α are related to the mass eigenstate fields ν_i by $\nu_\alpha = \Sigma_i U_{\alpha i} \nu_i$, where the $U_{\alpha i}$ are elements of a unitary matrix. In the limit of vanishing mixing angles ($U_{\alpha i} = \delta_{\alpha i}$) and a thermal background, the effective interaction of Eq. (43) gives the same mass shifts as obtained in Ref. [19]. For nontrivial mixing, however, in terms of the flavor fields Eq. (43) becomes

$$\mathcal{H}_I^{\rm eff} = \sqrt{2} G_F \sum_{\alpha,j} \overline{\nu}_\alpha J^\mu_{\nu_j} \gamma_\mu P_L \nu_\alpha + \sqrt{2} G_F \sum_{\alpha,\beta} \overline{\nu}_\alpha (U_{\alpha i} J^\mu_{\nu_i} U_{i\beta}) \gamma_\mu P_L \nu_\beta. \tag{45}$$

Considered as a matrix in flavor space, the quantity in parentheses in the second term of Eq. (45) contains off-diagonal elements. It is clear from our derivation that the presence of these off-diagonal terms derives from the fact that the background neutrinos are mass eigenstates, since they must be on shell. This origin of off-diagonal flavor space terms in the background potential due to neutrino-neutrino scattering was pointed out in Ref. [23].

In calculations of the effects of neutrino flavor oscillations in the supernova environment, the supernova core is typically treated as a stationary source of neutrinos free-streaming from a ''neutrinosphere,'' with the flux at the neutrinosphere being taken from large-scale numerical computations. Since the neutrinos forming the background also undergo flavor transformation, self-consistency between the oscillation probability and the background must be achieved. Following the framework of Ref. [23], such a calculation was carried out in Ref. [22] in the quantum mechanical picture of neutrino oscillations. This involved a rather complicated procedure involving a flavor basis density matrix to describe the neutrinos above the neutrinosphere. Casual inspection of the form of the second term in Eq. (37) would seem to make this kind of approach necessary. However, having shown that this term can plausibly be written in terms of a macroscopic mass basis current, we see that the self-consistency between neutrino background and oscillation probability is most easily achieved by working in the mass eigenstate basis.[8] Given effective interaction Hamiltonians like Eqs. (33) and (43), the oscillation probability can be computed (in any basis) as described in Secs. IV and V. We write Eq. (44) as

[8]It would seem reasonable to define neutrino flavor distribution functions in the relativistic limit. While *at the emission point* (i.e. the neutrinosphere) one could argue that these would be related to the mass basis distribution functions by $f_{\nu_i}(\mathbf{p}) = \Sigma_\alpha |U_{\alpha i}|^2 f_{\nu_\alpha}$, at points above the neutrinosphere the relation between these sets of distribution functions is rather complicated, due to the flavor/mass oscillations of free-streaming neutrinos in a background. The resulting absence of a simple connection between the macroscopic flavor and mass neutrino currents at arbitrary positions to plug into Eq. (45) makes working in the mass basis seem much more straightforward.

$$J^{\mu}_{\nu_i}(r)=\int \frac{E_{\mathbf{p}}^2\,dE_{\mathbf{p}}\,d(\cos\theta)d\phi}{(2\pi)^3}[f_{\nu_i}(E_{\mathbf{p}},\cos\theta,r)-f_{\bar{\nu}_i}(E_{\mathbf{p}},\cos\theta,r)]\frac{p^{\mu}}{E_{\mathbf{p}}}, \tag{46}$$

where $p^{\mu}=(E_{\mathbf{p}},\hat{\mathbf{p}}E_{\mathbf{p}})$, r is the radial position of a point above the neutrinosphere, and θ is the angle between the neutrino momentum and the radial direction at the point with radial position r. Since the neutrinos are free-streaming, the distribution functions at r can be expressed simply in terms of the ''known'' neutrino distribution functions at the neutrinosphere, e.g.:

$$f_{\nu_i}(E_{\mathbf{p}},\cos\theta,r)=\sum_{\nu_j} f_{\nu_j}(E_{\mathbf{p}},\cos\psi,R)\,P_{\nu_j\to\nu_i}(E_{\mathbf{p}},\cos\theta,r), \tag{47}$$

where R is the radius of the neutrinosphere, and ψ is the angle of the neutrino emission with respect to the radial direction at the emission point; this angle is related to θ by $\cos\psi=\sqrt{1-[(r/R)\sin\theta]^2}$. The dependence of the oscillation probabilities on path length (and the background encountered on a particular path) are implicit in the r,θ dependence. With an iterative procedure, self-consistency between the macroscopic neutrino currents and the oscillation probabilities should be achieved.

IV. CONSTANT BACKGROUND

Having constructed effective interaction Hamiltonians as described in the last section, we employ the neutrino effective Lagrangian

$$\mathcal{L}=\overline{\nu}[\gamma^{\mu}(i\partial_{\mu}-V_{\mu}P_L)-M]\nu, \tag{48}$$

where M is the mass matrix and V_{μ} is a uniform background potential matrix. We make no assumptions about the number of neutrino generations or the structure of the potential matrix (other than to keep in mind that it might be singular). The canonical anticommutation relations yield the equation satisfied by the Green's function $G(x,y)$,

$$[\gamma^{\mu}(i\partial_{\mu}-V_{\mu}P_L)-M]G(x,y)=\delta^4(x-y), \tag{49}$$

where $i\,G(x,y)\equiv\langle T\psi(x)\overline{\psi}(y)\rangle_0$. With our convention for the γ matrices it is convenient to define the 2×2 (in spinor space) matrices G_{IJ}, the ''chiral blocks'' of the Green's function. Specifically, G_{IJ} is the nonzero 2×2 submatrix of $P_I G P_J$, where I,J can take the values L,R.

In Sec. II we noted that, with the assumption of $V-A$ interactions, G_{LR} is the object of interest. We also saw in Sec. II that with the assumption of stationarity it is natural to Fourier transform the time variable while maintaining interest in the spatial dependence of the Green's function. Defining $J=M^{-1}G_{RR}$, from Eq. (49) we find

$$G_{LR}(\omega,\mathbf{x},\mathbf{y})=(\omega+i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})J(\omega,\mathbf{x},\mathbf{y}), \tag{50}$$

where $f(x,y)=\int(d\omega/2\pi)\,e^{-i\omega(x^0-y^0)}f(\omega,\mathbf{x},\mathbf{y})$. In the context of neutrino oscillation experiments we are interested in well-separated source and detector positions, so we ignore terms in J with more than one factor of $|\mathbf{x}-\mathbf{y}|$ in the denominator,

$$\begin{aligned}J(\omega,\mathbf{x},\mathbf{y})&=\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}J(\omega,\mathbf{p})\\&=\int_0^{\infty}\frac{du\,u^2}{(2\pi)^3}\int_0^{2\pi}d\phi\,\frac{1}{iu|\mathbf{x}-\mathbf{y}|}\\&\quad\times\Bigg[e^{iu|\mathbf{x}-\mathbf{y}|}J(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})\\&\quad-e^{-iu|\mathbf{x}-\mathbf{y}|}J(\omega,u,\hat{\mathbf{p}}=-\hat{\mathbf{r}})+\mathcal{O}\left(\frac{J}{u|\mathbf{x}-\mathbf{y}|}\right)\Bigg],\end{aligned} \tag{51}$$

where we have integrated the $\cos\theta$ integral by parts, and defined $u\equiv|\mathbf{p}|$ and $\hat{\mathbf{r}}\equiv(\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|$. It is evident that the two leading terms are azimuthally symmetric, and that their the sum is even in u. Furthermore, the Feynman boundary conditions should ensure that the two leading terms give equal contributions. We then have

$$J(\omega,\mathbf{x},\mathbf{y})\simeq\frac{1}{(2\pi)^2 i|\mathbf{x}-\mathbf{y}|}\int_{-\infty}^{\infty}du\,u\,e^{iu|\mathbf{x}-\mathbf{y}|}J(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}}). \tag{52}$$

From Eq. (49), we find that J obeys the momentum space equation

$$[\omega^2-|\mathbf{p}|^2-M^2-\omega V^0+\mathbf{p}\cdot\mathbf{V}+\boldsymbol{\sigma}\cdot(V^0\mathbf{p}-\omega\mathbf{V}+i\mathbf{V}\times\mathbf{p})]J(\omega,\mathbf{p})=1, \tag{53}$$

or $D(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})J(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})=1$. Detailed expressions for the spinor space elements of $D(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})$ for general orientation of $\mathbf{r}$ are not particularly illuminating. However, it is easy to formally express the spinor space elements of $J(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})$ in terms of the elements of D:

$$J^{11}(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})=[D^{11}-D^{12}(D^{22})^{-1}D^{21}]^{-1}, \tag{54}$$

$$J^{22}(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})=[D^{22}-D^{21}(D^{11})^{-1}D^{12}]^{-1}, \tag{55}$$

$$J^{12}(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})=-(D^{11})^{-1}D^{12}J^{22}, \tag{56}$$

$$J^{21}(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})=-(D^{22})^{-1}D^{21}J^{11}. \tag{57}$$

These results are valid without any relativistic limit assumptions. Given specific mass and potential matrices, one could solve explicitly for $J(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})$. To study neutrino oscillations, we then need $J(\omega,\mathbf{x},\mathbf{y})$, whose behavior is seen from Eq. (52) to be determined by the poles of $J(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})$ with the positive imaginary part (as determined by the Feynman boundary conditions).

A few general comments regarding these poles are in order. Consider for example J^{22}, which can be expressed

$$J^{22}(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})=(\det D^{11})[(\det\ D^{11})D^{22}-D^{21}(C^{11})^T D^{12}]^{-1}, \tag{58}$$

where $(C^{11})^T$ is the transpose of the matrix of cofactors of D^{11}. Since the diagonal elements of D^{11} and D^{22} are second order in u, $(\det D^{11})$ is of order $2n$ in u, where n is the number of neutrino generations; and overall the denominator of $J^{22}(\omega,u,\hat{\mathbf{p}}=+\hat{\mathbf{r}})$ will be a polynomial of order $4n$ in u. This is sensible in terms of a quasiparticle picture associated with the propagator: Each neutrino field, with two spin states each for particles and antiparticles, represents four states. For a single vacuum field the masses of these four states are degenerate; however, the presence of a parity and rotational invariance violating potential breaks this degeneracy.

Let us examine the simplifications that occur in the relativistic limit and with source and detector localization. In the relativistic limit the poles contributing to the integral in Eq. (52) take the form

$$u\simeq|\omega|-\frac{\tilde{m}^2}{2|\omega|}+i\epsilon, \tag{59}$$

with the Feynman boundary conditions imposed by giving the ‘‘masses’’ $\tilde{m}^2$ a small negative imaginary part. (There are also negative poles, with negative imaginary parts, that do not contribute to the integral; these factors each become $\simeq 2|\omega|$ when evaluated at the positive poles.) Furthermore, following the discussion of Sec. II regarding spatial localization, Eq. (52) takes the form

$$J(\omega,\mathbf{x},\mathbf{y})\simeq\frac{2|\omega|\,e^{i\omega\hat{\mathbf{L}}\cdot(\mathbf{x}-\mathbf{y})}}{4\pi|\mathbf{x}_S-\mathbf{y}_D|}\sum_j e^{-i(\tilde{m}_j^2/2|\omega|)|\mathbf{x}_S-\mathbf{y}_D|}\left[\left(u-|\omega|+\frac{\tilde{m}_j^2}{2|\omega|}\right)J\left(\omega,u,\hat{\mathbf{p}}=\frac{\omega}{|\omega|}\hat{\mathbf{L}}\right)\right]\Bigg|_{u\to|\omega|-(\tilde{m}_j^2/2|\omega|)}, \tag{60}$$

where the sum is over the poles with positive imaginary parts. We recall that ω is fixed by energy δ functions, to a positive value for neutrino oscillations and a negative value for antineutrino oscillations. Because the spatial localization sets $\hat{\mathbf{r}}=\pm\hat{\mathbf{L}}$ (with $\hat{\mathbf{L}}$ taken to be the third spatial direction), the matrix D takes the relatively simple spinor space form

$$\left[D\left(\omega,u,\hat{\mathbf{p}}=\frac{\omega}{|\omega|}\hat{\mathbf{L}}\right)\right]=\begin{pmatrix} d-\omega\left(1-\frac{u}{|\omega|}\right)V^0+\omega\left(\frac{u}{|\omega|}-1\right)V^3 & -\omega\left(1+\frac{u}{|\omega|}\right)(V^1-iV^2) \\ -\omega\left(1-\frac{u}{|\omega|}\right)(V^1+iV^2) & d-\omega\left(1+\frac{u}{|\omega|}\right)V^0+\omega\left(\frac{u}{|\omega|}+1\right)V^3 \end{pmatrix}, \tag{61}$$

where $d\equiv\omega^2-u^2-M^2$.

Next we examine the momentum space pole structure of J^{22} in the relativistic limit. In Eq. (55), since the residues of only the positive poles contribute, we can replace u by $|\omega|$ whenever it multiplies a component of V^μ, committing errors of only $\mathcal{O}(\omega V^0/\omega^2)$ or less with respect to other terms present. In that case, D_{21} vanishes and $J^{22}\to(D^{22})^{-1}$ with

$$D^{22}\to\omega^2-u^2-M^2-\frac{\omega}{|\omega|}2q\cdot V, \tag{62}$$

where $q=(|\omega|,\hat{\mathbf{L}}|\omega|)$ is the same as the neutrino momentum defined just below Eq. (4). We note that the denominator of J^{22} is now only of order $2n$ in u; thus in the relativistic case, two of the quasiparticle propagating states are projected out. This is because in the relativistic limit the spin states naturally coincide with the chiral states. This is confirmed by noting that in Eq. (55), for example ($\omega>0$ case), as $D^{21}\to 0$ as $u\to\omega$, half of the poles contributing to J^{22} come from $(\det D^{11})\to 0$. But for $u\to\omega$, $D^{11}\to\omega^2-u^2-M^2$. Thus these poles correspond to the vacuum masses; these are the right-handed particle states and left-handed antiparticle states whose masses are unaffected by the left-handed effective potential. As D^{21} reaches zero, the contribution of these poles vanishes completely.

For $|\omega||\mathbf{x}-\mathbf{y}|\gg 1$, Eq. (50) becomes

$$G_{LR}(\omega,\mathbf{x},\mathbf{y})\simeq\omega(1-\boldsymbol{\sigma}\cdot\hat{\mathbf{L}})J(\omega,\mathbf{x},\mathbf{y}). \tag{63}$$

Since we have chosen $\hat{\mathbf{L}}=(\mathbf{y}_D-\mathbf{x}_S)/|\mathbf{y}_D-\mathbf{x}_S|$ to coincide with the third spatial dimension, the only nonzero component of G_{LR} is G_{LR}^{22} which for neutrino oscillations is $G_{LR}^{22}(|\omega|;\mathbf{y},\mathbf{x})=2|\omega|J^{22}(|\omega|,\mathbf{y},\mathbf{x})$, where the spinor space indices are exhibited and the mass/flavor indices are suppressed (note that J^{21} vanishes), and for antineutrino oscillations is $G_{LR}^{22}(-|\omega|;\mathbf{x},\mathbf{y})=-2|\omega|J^{22}(-|\omega|,\mathbf{x},\mathbf{y})$. Thus we see that the Green’s function takes the required form of Eqs. (17).

The matrix $M^2+2q\cdot V$ (or $M^2-2q\cdot V$) of Eq. (62) is precisely the effective mass matrix $\tilde{M}^2$ appearing in the usual quantum mechanical model of neutrino (or antineutrino) oscillations. The effective mass matrix can be diagonalized by a unitary transformation, $\tilde{U}\tilde{M}^2\tilde{U}^\dagger=1$. Thus for neutrino oscillations, for example,

$$(D^{22})^{-1}_{\beta\alpha}=\tilde{U}_{\beta j}\tilde{U}^*_{\alpha j}(\omega^2-u^2-\tilde{m}_j^2+i\epsilon)^{-1}, \tag{64}$$

where $\tilde{U}_{\beta j}$ are the elements of $\tilde{U}$ and $\tilde{m}_j^2$ are the eigenvalues of $\tilde{M}^2$. Then Eq. (63) becomes, using Eq. (60),

$$G_{LR}^{\beta\alpha}(\omega,\mathbf{y},\mathbf{x})=-|\omega|(1-\boldsymbol{\sigma}\cdot\hat{\mathbf{L}})\frac{e^{i|\omega|\hat{\mathbf{L}}\cdot(\mathbf{y}-\mathbf{x})}}{4\pi|\mathbf{y}_D-\mathbf{x}_S|}\times\sum_j \tilde{U}_{\beta j}\tilde{U}^*_{\alpha j}e^{-i(\tilde{m}_j^2/2|\omega|)|\mathbf{y}_D-\mathbf{x}_S|}, \tag{65}$$

where we have assumed that the various conditions discussed in Sec. II are satisfied. Comparison of Eq. (65) with Eqs. (17), (24) shows that the oscillation probability derived here is precisely the same as that found in the usual quantum mechanical model. The antineutrino case works out in a similar manner.

V. NONUNIFORM BACKGROUND

In this section we consider the case in which the effective potential $V^\mu=V^\mu(\mathbf{x})$, that is, we allow it to vary in space (but not time). From Eqs. (65), (17), and (24), it is clear that in the constant potential case the portion of the Green's function comprising the oscillation amplitude obeys a Schrödinger-type equation, the same one used in the standard quantum mechanical picture. While one might think to simply replace the constant potential in this Schrödinger equation with a spatially varying one—thus arriving immediately at the standard result—we shall go back a little further in order to see what is being left out in the process.

In allowing for spatial variation in V^μ, Eqs. (50)–(52) are unchanged; but Eq. (53) becomes an integral equation, as $J(\omega,\mathbf{p})$ must be convolved with the momentum space dependence of V^μ. Since such equations are difficult to deal with nonperturbatively, in this section we take a different route of working with a partial differential equation in coordinate space. The coordinate space version of Eq. (53) is

$$\begin{aligned}[\omega^2+\nabla^2-M^2-\omega V^0(\mathbf{x})-i\mathbf{V}(\mathbf{x})\cdot\boldsymbol{\nabla}\\ -i\boldsymbol{\sigma}\cdot(V^0(\mathbf{x})\boldsymbol{\nabla}-i\omega\mathbf{V}(\mathbf{x})+i\mathbf{V}(\mathbf{x})\times\boldsymbol{\nabla})]J(\omega,\mathbf{x},\mathbf{y})\\ =\delta^3(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{66}$$

Unlike the case of an integral equation, to define the Green's function using this equation, we must also separately specify the boundary condition. We shall assume that the production region is localized to a region of adiabatically constant potential. Furthermore, since we expect the virtual particles to all have the same phase just after being produced, our boundary condition prescription will be that the Green's function J asymptotes to the constant potential Green's function on an infinitesimal sphere centered about $\mathbf{y}$.[9]

The form of Eq. (52), together with our experience in the vacuum and constant potential cases, suggests that in the relativistic limit $[M^2/2|\omega|^2\ll 1, V|\omega|/|\omega|^2\ll 1$ where we suppressed the matrix indices], we look for solutions of J of the form

$$J(\omega,\mathbf{x},\mathbf{y})=-\frac{e^{i|\omega||\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}F(\omega,\mathbf{x},\mathbf{y}). \tag{67}$$

With this substitution,

$$(\nabla^2+\omega^2)J=\delta^3(\mathbf{x}-\mathbf{y})e^{i|\omega||\mathbf{x}-\mathbf{y}|}F-\frac{2|\omega|\,e^{i|\omega||\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\left[\frac{1}{2|\omega|}\nabla^2F+i(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}F)-\frac{1}{|\omega||\mathbf{x}-\mathbf{y}|}(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}F)\right], \tag{68}$$

$$\boldsymbol{\nabla}J=-\frac{2|\omega|\,e^{i|\omega||\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\left[\frac{i\hat{\mathbf{r}}}{2}F+\frac{1}{2|\omega|}\boldsymbol{\nabla}F-\frac{\hat{\mathbf{r}}}{2|\omega||\mathbf{x}-\mathbf{y}|}F\right], \tag{69}$$

where as before $\hat{\mathbf{r}}\equiv(\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|$. Requiring the first term on the right-hand side of Eq. (68) to cancel the δ function in Eq. (66) gives a boundary condition on F, namely (restoring flavor indices)

$$F^{\beta\alpha}(\omega,\mathbf{x},\mathbf{y})|_{\mathbf{x}\to\mathbf{y}}=\delta^{\beta\alpha}. \tag{70}$$

Aside from this boundary condition, we are interested in well-separated $\mathbf{x}$ and $\mathbf{y}$ (specifically $|\omega||\mathbf{x}-\mathbf{y}|\gg 1$), so that we may ignore the last term of Eqs. (68) and (69). Then Eq. (66) becomes

$$\begin{aligned}i(\hat{\mathbf{r}}\cdot\boldsymbol{\nabla}F)+\frac{1}{2|\omega|}\nabla^2F-\frac{1}{2|\omega|}[M^2+\omega V^0-|\omega|(\hat{\mathbf{r}}\cdot\mathbf{V})\\ -\boldsymbol{\sigma}\cdot(V^0|\omega|\hat{\mathbf{r}}-\omega\mathbf{V}+i|\omega|\mathbf{V}\times\hat{\mathbf{r}})]F+\mathcal{O}\left(\frac{V|\omega|}{|\omega|^2}|\boldsymbol{\nabla}F|\right)=0,\end{aligned} \tag{71}$$

where in accordance with the relativistic condition $V|\omega|/|\omega|^2\ll 1$, we will neglect the terms represented by $\mathcal{O}(V|\omega||\boldsymbol{\nabla}F|/|\omega|^2)$ in comparison with the first term of Eq. (71).

One can distinguish three cases: (1) $|\boldsymbol{\nabla}F|\gg\epsilon F$, where $\epsilon=V^0+\tilde{M}^2/(2|\omega|)$ and $\tilde{M}^2$ denotes the largest mass matrix eigenvalue squared; (2) $|\boldsymbol{\nabla}F|\sim\epsilon F$; and (3) $|\boldsymbol{\nabla}F|\ll\epsilon F$. One can argue that case (1) is not interesting since all terms leading to flavor mixing are rendered negligible. Case (3) is also not of present interest because one can argue using Eq. (71) that it violates our relativistic assumption. In case (2),

[9]Note that one must match more than just the limiting singularity of the Green's function at $\mathbf{x}=\mathbf{y}$ to define a unique solution.

$|\nabla^2 F/(2|\omega|)|$ can be neglected compared with $|\nabla F|$, provided that $|\nabla V^0|/\epsilon^2 \lesssim 1$. Writing $F' \equiv \hat{\mathbf{r}} \cdot \nabla F$, Eq. (71) becomes

$$iF' + \frac{1}{2|\omega|} D(|\omega|,\mathbf{x})F = 0, \tag{72}$$

where the spinor space elements of $D(|\omega|,\mathbf{x})$ for the particular case when $\mathbf{x}$ lies along the third spatial axis $\hat{\mathbf{L}}$ are given by Eq. (61) with $V \to V(\mathbf{x})$ and $u \to |\omega|$.

Since Eq. (63) holds under the assumptions of case (2), together with the spatial localization of the source and detector, we see that G_{LR}^{22} is the only nonvanishing component. Hence, from Eqs. (72), (67), (17), and (24), we find that the neutrino oscillation amplitude H obeys the Schrödinger equation

$$iH' = \frac{1}{2|\omega|}[M^2 + 2q \cdot V(\mathbf{x})]H, \tag{73}$$

where $q = (|\omega|, \hat{\mathbf{L}}|\omega|)$. Similarly, in the case of antineutrino oscillations $(\omega < 0)$, the oscillation amplitude obeys

$$i\overline{H}' = \frac{1}{2|\omega|}[M^2 - 2q \cdot V(\mathbf{x})]\overline{H}. \tag{74}$$

Before concluding this section, a remark regarding the boundary conditions is in order. Note that F of Eq. (72) satisfying the boundary condition Eq. (70) is in general different from F satisfying Eq. (66) with its associated boundary condition [described just below Eq. (66)]. In particular, although Eq. (72) is valid naively only far away from $\mathbf{y}$, as we threw out the last terms of Eqs. (68) and (69), we still insisted on the boundary condition Eq. (70) at $\mathbf{y}$ to be the same as the boundary condition that would have been used for the exact equation. To justify this, we must show that the terms that we threw out are negligible even near the origin. We can argue this by noting that for case (2), we are already assuming $|\nabla V^0|/\epsilon^2 \lesssim 1$ which turns out to imply (by expanding the potential to linear order in the Taylor series about $\mathbf{y}$) that in the relativistic limit the fractional variation of V^0 is much smaller than 1 until $|\omega(\mathbf{x}-\mathbf{y})| \gg 1$ (after which the terms proportional to $1/|\mathbf{x}-\mathbf{y}|$ that we threw out are negligible). That means that the potential can be treated as a constant until the terms proportional to $1/|\mathbf{x}-\mathbf{y}|$ become negligible. This implies that one can place the boundary condition for the varying potential case on a sphere (centered about $\mathbf{y}$) on which Eq. (72) is valid using the solution to the constant potential case. As we saw in the last section, since the exact solution to the constant potential case on this sphere (with the appropriate boundary condition) is, up to relativistically suppressed terms, the same as the solution obtained by Eq. (72) with Eq. (70), we can just set the boundary condition for the varying potential case using Eq. (70) as well. Note, however, that since our argument depends on the adiabaticity of the potential near the virtual particle production point, in other situations, one may need to be more cautious with the boundary conditions.

Thus under appropriate conditions the results of the usual simplified picture are confirmed, including the boundary condition $H^{\beta\alpha}(\omega,\mathbf{x},\mathbf{y})|_{\mathbf{y}\to\mathbf{x}} = \delta^{\beta\alpha}$.

VI. CONCLUSION

Starting from quantum field theory (QFT), we have defined a physically meaningful flavor oscillation probability, determined that portion of the neutrino propagator that comprises the oscillation amplitude, and derived the ''Schrödinger equation'' for that amplitude in the presence of spatially varying background matter. As expected, the ''Schrödinger equation'' really corresponds to a time-independent one since its derivation depends on the time independence of the effective potential. In fact, the usual quantum mechanical approach is only really suited to problems in which oscillations occur in space only (that is, stationary systems like that studied here) or time only (e.g. the thermal bath in the early universe). While we have assumed the stationary case here, the basic framework could also be used to study oscillations in space in the presence of a time-dependent background. Ultimately, the description of flavor oscillating neutrinos in space and time in more general systems—i.e. those that do not lend themselves to interpretation in terms of a ''source'' and ''detector''—would require a formulation in terms of density matrices (cf. Ref. [23]) or Wigner functions (cf. Ref. [16]).

For situations that can be interpreted in terms of a ''source'' and ''detector,'' we have also reviewed the conditions under which QFT will be useful in describing the neutrino oscillation process.[10] As long as we are looking in the regime in which the virtual neutrino goes on shell (rendering any propagator radiative corrections to be negligible or be merely a constant shift), the many body aspect of QFT is rendered irrelevant. In that case, only the production/detection vertex structure and the spins of the neutrinos are missing from the usual quantum mechanical treatment. These, in general, are less difficult to accommodate in the quantum mechanical treatment than the many body effects. Still, we believe them to be more straightforwardly accommodated in the quantum field theoretical treatment. For weak interactions, the chiral nature of the interactions combined with the relativistic nature of the on shell neutrinos suppresses all but one spin degree of freedom. Finally, the smallness of the neutrino mass splittings as well as the neutrinos going on shell allows one to factor out the production/detection part of the neutrino scattering process from the ''oscillation'' part, reducing the problem to the usual quantum mechanical system involving a single spinless particle.

To state this another way, we have argued that in the context of stationary systems, the quantum field theoretic formulation used in this work is not of much use except when one or more of the following is true: the on-shell neu-

[10]Many of these conditions were noted in Refs. [10–15].

trino momentum is nonrelativistic, the production or detection vertices are non-chiral, the external particle wave packets vary appreciably about the pole value (value of the wave packet evaluated at the neutrino momenta) over momentum variations of the order of the inverse source-detector distance $1/L$, or the effective mass splittings (determined by the effective potential including the background matter contributions) are large compared to the spread in the momenta of the external production/detection particles.

When the neutrinos are non-relativistic or the interactions are not chiral, more than one spin contributes per amplitude. In that case the usual quantum mechanical treatment must be modified to incorporate the effects due to the various spin components. In particular, in this case an oscillation probability cannot be determined apart from the production and detection processes since two neutrino spin contributions are summed before the amplitude is squared. This is, of course, automatically accounted for in a quantum field theoretic treatment. Also, if the wave packet varies appreciably about the pole value when the momentum is varied by $\mathcal{O}(1/L)$, the wave functions of the external particles creating and absorbing the virtual neutrino do not simply factorize out of the oscillation probability amplitude. Reference [15] is a study of one such situation. They find that the oscillation amplitude can exhibit a novel ''plane-wave'' behavior. Just as with most effects, this probably can also be accounted for in the quantum mechanical formulation, but it is much easier in the quantum field theoretical treatment used in this paper. The involvement of the details of the external particles' wave function in determining the oscillation probability also applies when the effective mass splitting is much larger than the momentum spread in the external wave packets, but in such cases, it is not clear whether any neutrino oscillations can be observed in general because of the strong suppression of the amplitudes (however, see for example Ref. [15]).

We here make a few comments regarding Majorana vs Dirac neutrinos. If the neutrinos were Majorana instead of Dirac, the only general arguments that would change are those dependent upon the existence of a right-handed neutrino. Although we couched the mathematics in the Dirac spinor formalism, owing to the assumption of chiral nature of neutrino interactions and the relativistic limit, none of our general arguments depended upon the existence of a right handed neutrino. Hence, our conclusions for the relativistic limit are also valid for Majorana neutrinos. (Formulas for ''neutrinos'' apply to negative helicity Majorana neutrinos, while those for ''antineutrinos'' apply to positive helicity Majorana neutrinos.)

From our justification of the ''Schrödinger equation'' for stationary, flat spacetime systems, we expect that the heuristic ansatz used in Ref. [24] for studying neutrino oscillations in a stationary curved spacetime to be valid to the extent that the flat spacetime treatment is valid. In a nonstationary curved spacetime, in addition to the time dependence of the potential, there may arise extra complications of using the S-matrix formalism due to the nontrivial Bogoliubov transformations of the asymptotic states. This also deserves further investigation.

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APPENDIX: BOX QUANTIZATION AND LOCALIZATION

In Sec. II we discussed the crucial role that wave packets play in the localization inherent in neutrino oscillation experiments. Studies of vacuum oscillations in quantum field theory typically begin with some kind of coordinate space description of the external particles, with the connection to the underlying momentum space wave packets not explicitly displayed. Since normalization in quantum field theory is generally fixed at the level of free-particle momentum eigenstates, fully normalized event rates cannot be computed unless the connection to momentum space is made. The usual practice in the literature has been to ignore this step, being satisfied with the identification of the factor called the ''oscillation amplitude'' in the simple quantum mechanical picture. The complete connection to momentum space is presumably a straightforward exercise, though perhaps tedious in the context of general wave packets. However, as shown in the text, the box wave packet allows a simple, reasonable approximation. For typical microscopic processes, a standard trick used to relate the S matrix to rates and cross sections is to employ the fiction of a box with quantized states, which reproduces the box wave packet results. For pedagogical purposes, we here demonstrate that the use of a similar fiction—separate boxes confining the external particles at the source and detector—leads cleanly and directly to an event rate in the form of Eq. (1). In this heuristic approach the boxes serve the double duty of imposing the localization of source and detector necessary to the existence of oscillations, and the normalization that gives an absolute event rate.

As an illustration we will consider a particular neutrino oscillation process. The first vertex of the process we consider involves the scattering of a charged lepton of flavor α off a proton in a source region of volume V_S, producing a final state neutron confined to the source region and a virtual neutrino that propagates over all space. At the second vertex, the virtual neutrino interacts with a neutron confined to a detection region of volume V_D, producing a proton and charged lepton of flavor β confined to the detection region. The free Lagrangian is taken to be

$$L_0=\sum_s \int_{V_S} d^3\mathbf{x}\, \overline{\psi_s}(i\gamma^\mu\partial_\mu-m)\psi_s + \sum_d \int_{V_D} d^3\mathbf{x}\, \overline{\psi_d}(i\gamma^\mu\partial_\mu-m)\psi_d + \sum_i \int_{\text{all space}} d^3\mathbf{x}\, \overline{\psi_i}(i\gamma^\mu\partial_\mu-m)\psi_i, \tag{A1}$$

where the index s runs over the particles confined to the source (α,p,n), the index d runs over the particles confined to the detector (β,p,n), and the index i runs over the neutrino mass eigenstates. The box localization is really a result of complicated classical potentials preparing the initial state. The expansions of the source and detector particle fields are[11]

$$\psi_{s,d}(x)=\frac{1}{\sqrt{V_{S,D}}}\sum_{\sigma}\sum_{\mathbf{p}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}[a_{\mathbf{p},\sigma,(s,d)}u_{\mathbf{p},\sigma,(s,d)}e^{-ip\cdot x}+b^{\dagger}_{\mathbf{p},\sigma,(s,d)}v_{\mathbf{p},\sigma,(s,d)}e^{ip\cdot x}], \tag{A2}$$

where the momentum sum is over the discrete momenta arising from the application of periodic boundary conditions to the box, the index σ is over the spin states, and the commutation relations of the creation/annihilation operators are of the form $[a_{\mathbf{p},\sigma,s},a^{\dagger}_{\mathbf{p}',\sigma',s'}]=\delta_{\mathbf{pp}'}\,\delta_{\sigma\sigma'}\,\delta_{s,s'}$. In contrast, the free neutrino fields have a continuous momentum spectrum,

$$\psi_i(x)=\sum_{\sigma}\int\frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}[a(\mathbf{p},\sigma,i)u(\mathbf{p},\sigma,i)e^{-ip\cdot x}+b^{\dagger}(\mathbf{p},\sigma,i)v(\mathbf{p},\sigma,i)e^{ip\cdot x}] \tag{A3}$$

with $[a(\mathbf{p},\sigma,i),a^{\dagger}(\mathbf{p}',\sigma',i')]=\delta^3(\mathbf{p}-\mathbf{p}')\,\delta_{\sigma\sigma'}\,\delta_{ii'}$.

The interaction Lagrangian relevant to the first vertex of the process described above is

$$L_I=-g\int_{V_S}d^3\mathbf{x}\,\overline{\psi}_{\alpha}\gamma^{\mu}(1-\gamma_5)\nu_{\alpha}\overline{\psi}_p\gamma_{\mu}(1-g_A\gamma_5)\psi_n+\text{H.c.}, \tag{A4}$$

where $g\equiv G_F\cos\theta_c/\sqrt{2}$, G_F is the Fermi constant, θ_c is the Cabibbo angle, and $g_A\approx 1.26$ is the neutron-proton axial vector coupling. A similar interaction Lagrangian describes the interaction in the detector. The relation between the flavor fields and mass eigenstate fields is $\nu_{\alpha}=\Sigma_i U_{\alpha i}\psi_i$, where the $U_{\alpha i}$ are elements of a unitary matrix. The amplitude for the neutrino production/detection process is

$$\begin{aligned}T_{\alpha\beta}=&-g^2\int_{-\infty}^{\infty}dx^0\int_{-\infty}^{\infty}dy^0\int_{V_S}d^3\mathbf{x}\\&\times\int_{V_D}d^3\mathbf{y}\frac{e^{-ip\cdot x}}{\sqrt{V_S}\sqrt{2E_{\mathbf{p}}}}\frac{e^{ip'\cdot x}}{\sqrt{V_S}\sqrt{2E_{\mathbf{p}'}}}\\&\times\frac{e^{-ik\cdot x}}{\sqrt{V_S}\sqrt{2E_{\mathbf{k}}}}\frac{e^{-il\cdot y}}{\sqrt{V_D}\sqrt{2E_{\mathbf{l}}}}\frac{e^{il'\cdot y}}{\sqrt{V_D}\sqrt{2E_{\mathbf{l}'}}}\\&\times\frac{e^{ik'\cdot y}}{\sqrt{V_D}\sqrt{2E_{\mathbf{k}'}}}[\overline{u}(\mathbf{l}')\gamma^{\mu}(1-g_A\gamma_5)u(\mathbf{l})]\\&\times[\overline{u}(\mathbf{k}')\gamma_{\mu}(1-\gamma_5)(iG^{\beta\alpha}(y,x))\gamma_{\nu}(1-\gamma_5)u(\mathbf{k})]\\&\times[\overline{u}(\mathbf{p}')\gamma^{\nu}(1-g_A\gamma_5)u(\mathbf{p})].\end{aligned} \tag{A5}$$

Here $\mathbf{k}$ and $\mathbf{k}'$ are respectively the initial and final lepton momenta, $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final source nucleon momenta, and $\mathbf{l}$ and $\mathbf{l}'$ are the initial and final detector nucleon momenta. The propagator or Green's function, $iG^{\beta\alpha}(y,x)=\langle T\{\nu_{\beta}(y)\overline{\nu}_{\alpha}(x)\}\rangle_0$, with $T\{\}$ and $\langle\rangle_0$ denoting a time-ordered product and vacuum expectation value, respectively, can be expressed in momentum space as

$$G^{\beta\alpha}(y,x)=\int\frac{d^4s}{(2\pi)^4}e^{-is\cdot(y-x)}G^{\beta\alpha}(s). \tag{A6}$$

Integration over x^0, y^0, and s^0 in Eq. (A5) fixes $|s^0|=|E_{\mathbf{k}}+E_{\mathbf{p}}-E_{\mathbf{p}'}|$ and yields a factor $(2\pi)^2\delta(E_{\mathbf{k}}+E_{\mathbf{p}}+E_{\mathbf{l}}-E_{\mathbf{k}'}-E_{\mathbf{p}'}-E_{\mathbf{l}'})$.

We let V_S and V_D be cubes of sides L_S and L_D, centered on $\mathbf{x}_S$ and $\mathbf{y}_D$, respectively, and employ the approximation of Eq. (21). Because of the $V-A$ lepton currents, the relevant block of the Green's function is G_{LR}. Suppose that this block of the propagator takes the form of Eq. (17), with $H^{\beta\alpha}$ having only flavor indices. Changing variables to $\mathbf{x}'=\mathbf{x}-\mathbf{x}_S$ and $\mathbf{y}'=\mathbf{y}-\mathbf{y}_D$, the amplitude in Eq. (A5) can be expressed

$$\begin{aligned}T_{\alpha\beta}=&\frac{ig^2\,\delta(E_{\mathbf{k}}+E_{\mathbf{p}}+E_{\mathbf{l}}-E_{\mathbf{k}'}-E_{\mathbf{p}'}-E_{\mathbf{l}'})}{2(V_S)^{3/2}(V_D)^{3/2}|\mathbf{y}_D-\mathbf{x}_S|\left(\prod_s\sqrt{2E_s}\right)\left(\prod_d\sqrt{2E_d}\right)}\\&\times e^{i(\mathbf{k}+\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}_S}e^{i(\mathbf{l}-\mathbf{l}'-\mathbf{k}')\cdot\mathbf{y}_D}H^{\beta\alpha}(E_{\mathbf{q}},\mathbf{y}_D,\mathbf{x}_S)\\&\times\left[\prod_I\Delta(u^I,L_S)\Delta(v^I,L_D)\right][\overline{u}(\mathbf{l}')\gamma^{\mu}(1-g_A\gamma_5)u(\mathbf{l})]\\&\times[\overline{u}(\mathbf{k}')\gamma_{\mu}(1-\gamma_5)u(\mathbf{q})^-][\overline{u}^-(\mathbf{q})\gamma_{\nu}(1-\gamma_5)u(\mathbf{k})]\\&\times[\overline{u}(\mathbf{p}')\gamma^{\nu}(1-g_A\gamma_5)u(\mathbf{p})],\end{aligned} \tag{A7}$$

where the index I is over the three momentum directions, $\mathbf{u}\equiv\mathbf{k}+\mathbf{p}-\mathbf{p}'-\mathbf{q}$, $\mathbf{v}\equiv\mathbf{l}-\mathbf{l}'-\mathbf{k}'+\mathbf{q}$, $\Delta(w,a)\equiv(2/w)\sin(wa/2)$, and $\mathbf{q}=E_{\mathbf{q}}\hat{\mathbf{L}}$.

An event rate is obtained by squaring the amplitude; interpreting one of the energy δ functions as $T/(2\pi)$, where T is a time interval; and dividing by T. At this stage we will

[11]Since we are taking momentum quantization ''seriously'' in this approach, it is convenient to adopt slightly different normalization conventions from the ones used in the main text. The differences are easily deduced from the explicit expressions for the field expansions and the commutation relations given above.

also make the approximation of a continuum of states in the source and detector, i.e. take $(L_S/2),(L_D/2)$ to be large. In this limit $\Delta(w,a)\rightarrow(2\pi)\delta(w)$, and $[\Delta(w,a)]^2 \rightarrow[(2/w)\sin(wa/2)|_{w\rightarrow 0}](2\pi)\delta(w)=2\pi a\,\delta(w)$. Finally, in the continuum limit there is a phase space factor of the form $d^3\mathbf{p}\,V/(2\pi)^3$ for each final state particle. The event rate obtained from Eq. (A7) is then

$$\begin{aligned} d\Gamma_{\alpha\beta}= & \frac{g^4\,\delta(E_{\mathbf{k}}+E_{\mathbf{p}}+E_{\mathbf{l}}-E_{\mathbf{k}'}-E_{\mathbf{p}'}-E_{\mathbf{l}'})}{4(2\pi)^4 V_S L^2\left(\prod_s 2E_s\right)\left(\prod_d 2E_d\right)} \\ & \times|H^{\beta\alpha}(E_{\mathbf{q}},\mathbf{y}_D,\mathbf{x}_S)|^2\delta^3(\mathbf{p}+\mathbf{k}-\mathbf{p}'-\mathbf{q}) \\ & \times\delta^3(\mathbf{l}+\mathbf{q}-\mathbf{l}'-\mathbf{k}')\,d^3p'\,d^3l'\,d^3k' \\ & \times\sum_{\text{spins}}[|\overline{u}(\mathbf{l}')\gamma^\mu(1-g_A\gamma_5)u(\mathbf{l}) \\ & \times\overline{u}(\mathbf{k}')\gamma_\mu(1-\gamma_5)u(\mathbf{q})|^2|\overline{u}(\mathbf{q})\gamma_\nu(1-\gamma_5)u(\mathbf{k}) \\ & \times\overline{u}(\mathbf{p}')\gamma^\nu(1-g_A\gamma_5)u(\mathbf{p})|^2]. \end{aligned} \tag{A8}$$

Using standard plane wave methods to compute the neutrino production rate and cross section factors in Eq. (1), it is easy to verify that Eq. (A8) is equivalent to Eq. (1), with $P_{\nu_\alpha\rightarrow\nu_\beta}=|H^{\beta\alpha}(E_{\mathbf{q}},\mathbf{y}_D,\mathbf{x}_S)|^2$. As noted in the text, explicit computations yield the oscillation probabilities found in the usual quantum mechanical model.

Equation (A8) represents the rate per source proton, source charged lepton, and detector proton. To get the total experimental rate it is necessary to sum over these source and detector particles. Employing the usual classical distribution function, for uniform spatial distribution the number of particles of type x is

$$N_x=V\int\frac{d^3\mathbf{p}}{(2\pi)^3}f_x(\mathbf{p}), \tag{A9}$$

so that the total event rate is

$$\begin{aligned} d\Gamma_{\text{exp}}= & \int V_S\frac{d^3\mathbf{p}}{(2\pi)^3}f_p(\mathbf{p})\,V_S\frac{d^3\mathbf{k}}{(2\pi)^3} \\ & \times f_\alpha(\mathbf{k})\,V_D\frac{d^3\mathbf{l}}{(2\pi)^3}f_p(\mathbf{l})\,d\Gamma_{\alpha\beta}. \end{aligned} \tag{A10}$$

We note that one of the factors of V_S cancels, leaving one factor each of V_S and V_D. These remaining two volume factors are replaced by $d^3\mathbf{x}_S$ and $d^3\mathbf{y}_D$, and integrations over these source and detector positions are performed. While we have explicitly shown how this works out for our particular chosen process, for any neutrino production/detection process one factor of V_S and V_D will remain at the end.

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